2704. Proposed by WILLIAM HOOVER, Columbus, Ohio.

A particle moves from rest under gravity down the arc of a parabola with the axis vertical and concavity upward; express the time to the vertex in terms of an elliptic integral of the second kind.

2705. Proposed by PAUL CAPRON, U. S. Naval Academy.

Find the area of a loop of the trochoid

$$x = \tfrac{1}{3}a(3\phi - \pi \sin \phi), \qquad y = \tfrac{1}{3}a(3 - \pi \cos \phi).$$

2706. Proposed by HARRIS F. MACNEISH, New York City.

Through a given point draw a straight line cutting a given straight line and a given circle, such that the part of the line between the given point and the given line may be equal to the part within the given circle.

2707. Proposed by S. A. COREY, Albia, Iowa.

Let a, b and c be the vector sides of a triangle. Construct another triangle with vector sides r, s and t, where

$$r = ma - dnb, \qquad s = na + (m + en)b.$$

Then prove that

$$(m^2 + emn + dn^2)(a^2 + eab \cos (ab) + db^2) = r^2 + ers \cos (rs) + ds^2,$$

where d, e, m and n are any scalar quantities; a, b, r and s are the tensors, or lengths, of the sides a, b, r and s, respectively; and cos (rs) is the cosine of the angle between r and s when placed coinitially.

2708. Proposed by WILLIAM HOOVER, Columbus, Ohio.

A uniform plank of length $2a$ and thickness $2h$ rests in equilibrium on a fixed rough horizontal cylinder of radius c, so that a vertical plane containing the dimension $2a$ and the center of gravity of the plank is at right angles to the axis of the cylinder; find the period of a complete small oscillation of the plank.

SOLUTIONS OF PROBLEMS.

492. (Algebra). Proposed by ARTEMAS MARTIN, Washington, D. C.

If two numbers, A and B, $B > A$, be selected at random, what is the probability that they have no common divisor?

SOLUTION BY WARREN WEAVER, Throop College of Technology, Pasadena, Cal.

The following treatment of this question is due to Tchebycheff, and has been taken from Fisher's *Mathematical Theory of Probability.*

If $p_2, p_3, \cdots, p_n$ denote respectively the probability that each of the primes $2, 3, 5, \cdots, n$ is not a common factor of A and B, then the probability that no prime number is a common factor is:

$$P = p_2 \cdot p_3 \cdot p_5 \cdot \cdots \cdot p_n \cdot \cdots \cdot \text{ad. inf.}$$

This follows from the multiplication theorem and the fact that the sequence of the prime numbers is infinite.

By dividing any integer by the prime n we obtain besides the quotient a certain remainder that must be one of the following numbers, namely,

$$0, 1, 2, 3, \cdots, (n - 1).$$

Each of the remainders may be considered as a possible event. The probability of obtaining 0 as a remainder is then $1/n$. This same quantity is the probability that n is a factor of B. The probability that both A and B *are* divisible by n is therefore $1/n^2$. The probability that the numbers A and B *do not* both have the prime factor n is then:

$$p_n = 1 - 1/n^2.$$

Hence,

$$P = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right) \cdots \text{ ad. inf.} \tag{1}$$

This infinite product is shown to have the value:

$$P = \frac{6}{\pi^2} = 0.60793\cdots.$$

It seems that there may perhaps be in this method a slight fallacy, however. For it considers the probability that two numbers A and B $(A < B)$ do not each have the factor n, where n is any one of the infinite sequence of prime numbers. However large A be, it is finite, and there are therefore an infinite number of primes larger than A none of which, obviously, can be a factor of both A and B. It would seem, then, that after a certain finite point the terms of the infinite product (1) should all be unity. The point at which the terms should begin to be unity would depend in some complicated and unknown way upon the value of A—unknown, of course, because we do not have an analytical expression for the rth prime, to say nothing of an expression for the largest prime smaller than a given number.

If we consider a slightly different example this becomes clearer. Suppose we have given a positive integer A. What is the probability that the proper fraction A/B is in its lowest terms, B being chosen at random to have any positive integral value larger than or equal to A? By the previous reasoning it will be:

$$p = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right) \cdots \left(1 - \frac{1}{s^2}\right),$$

where $s = A$ if A be prime or s is the largest prime $< A$ if A be not prime. For example, when $A = 15$

$$p = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)\left(1 - \frac{1}{7^2}\right)\left(1 - \frac{1}{11^2}\right)\left(1 - \frac{1}{13^2}\right) = 0.61807,$$

a value which differs from P by only 1.67 per cent.

Concerning the discussion here given to the result of Tchebycheff it is worth while to note that one may be led to incorrect intuitive conclusions because of the fact that we are unable to conceive of the process of taking a positive integer "at random." We are likely to think of taking one out of the interval from 0 to, say, 10^{50}, considering it from the biased point of view of practical experience exceedingly unlikely that we will get an integer from beyond this interval; while, as a matter of fact, the probability of obtaining by chance a positive integer within this interval is, of course, infinitely small as compared to the probability of the number coming from beyond this interval.

Since, therefore, when A is picked at random the probability that it will be less than, say, 100 is very small indeed and since the result $6/\pi^2$ is a very close approximation to the correct result for all cases when A turns out to be greater than 100 we may conclude that it is exceedingly likely that the result $6/\pi^2$ is close to the true probability. A more exact statement or calculation of the *a priori* probability of this event seems beyond the possibilities of the theory.

Remarks by Roger A. Johnson, St. Paul, Minnesota.

This problem is known in the literature of the science of probability as Tchebycheff's Problem.

The chance that an integer A has as factor a given prime m is $1/m$. The chance that both A and B, two integers, contain m as a factor is $1/m^2$. The chance that they have not this factor is $p_m = 1 - (1/m^2)$. The chance that they have no common prime factors is

$$P = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)\left(1 - \frac{1}{7^2}\right)\left(1 - \frac{1}{11^2}\right) \cdots = 6/\pi^2,$$

a result evaluated by Euler.

Some years ago, I observed this fact stated in a brief note in the *Philosophical Magazine*, and made an experimental verification. About 200 numbers were obtained, having three to five digits each, the last digits being distributed uniformly among the values 0, 1, 2, $\cdots$, 9. These were placed in a box, drawn two at a time, and compared. When all were drawn, they were

replaced, mixed thoroughly, and the process repeated. It was previously decided to stop at the end of 600 trials. Throughout the work, the ratio of relatively prime pairs to total trials oscillated near the value .6, and at the conclusion, the number of relatively prime pairs was 364, a value surprisingly close to the correct theoretical value 364.8. Of course, an experiment involving, say, 10,000 trials would be more satisfying, but this result is perhaps not without interest.

493 (Algebra). Proposed by ALBERT BABBITT, University of Nebraska.

Determine the coefficients b, c, d of the equation $x^3 + bx^2 + cx + d = 0$ so that they shall be roots of the same equation. [From *Supplemento a Periodico di Matematica.*]

SOLUTION BY H. S. UHLER, Yale University.

Since the coefficient of x^3 is $+ 1$ and as b, c, d are to be roots, we have the following conditions:

$$- b = b + c + d \ (1), \qquad c = bc + cd + db \ (2), \qquad - d = bcd \ (3).$$

Equation (3) gives

$$d = 0 \ (4), \qquad \text{or} \qquad bc = - 1 \ (5).$$

Combining (4) with (1) and (2) we get

$$2b + c = 0 \ (1'), \qquad bc - c = 0 \ (2').$$

Combining (5) with (1) and (2) we find $2b^2 + bd - 1 = 0$ (1″) and $b^2d - b -$.

Equations (1′) and (2′) lead at once to the two pairs of values $b = 0$, $c = 0$, and $b = 1$. $c = - 2$.

The factors of (2″) give $b = 1$ and $d = (b + 1)^{-1}$.

Substituting $b = 1$ in (5) and (1″) we find respectively $c = - 1$ and $d = - 1$. Replacing d by $(b + 1)^{-1}$ in (1″) gives

$$2b^3 + 2b^2 - 1 = 0. \qquad (6)$$

The discriminant of (6) is positive so that this cubic has two complex roots and one positive real root. These roots may be expressed as

$$b = \tfrac{1}{6}[(46 + 6\sqrt{57})^{1/3} + (46 - 6\sqrt{57})^{1/3}] - \tfrac{1}{3}.$$

The approximate value of the real root, b_1, is

$$b_1 = + 0.565,197,717,38.$$

Accordingly, the complex roots are approximately

$$b = \tfrac{1}{2}[- (b_1 + 1) + i\sqrt{(b_1 + 1)(3b_1 - 1)}],$$

or

$$b_2 = - 0.782,598,858,69 + i0.521,713,717,94,$$

$$b_3 = - 0.782,598,858,69 - i0.521,713,717,94.$$

The corresponding values of c and d may be found by the aid of (5) and $d = (b_1 + 1)^{-1}$, respectively. Finally, collecting all the results

b	c	d
0	0	0
1	-2	0
1	-1	-1
b_1	$-b_1^{-1}$	$(b_1 + 1)^{-1}$
b_2	$-b_2^{-1}$	$(b_2 + 1)^{-1}$
b_3	$-b_3^{-1}$	$(b_3 + 1)^{-1}$

Also solved by J. E. ROWE and the Proposer.

524 (Geometry). Proposed by NORMAN ANNING, Somewhere in France.

Many railways use as "easement curve" the cubic parabola. If points on such a curve are named by their distances measured along the curve from the point of inflection ("flat end")